Math 217.002 F25

Quiz 13 – Solutions

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1. Complete[*] the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose V and W are vector spaces. A *linear transformation* $T : V \to W$ is ...

   Solution: A function $T : V \to W$ such that for all $u, v \in V$ and all scalars α, β (from the underlying field $\mathbb{F}$),

   $$T(\alpha u + \beta v) = \alpha\, T(u) + \beta\, T(v).$$

   Equivalently: $T(u + v) = T(u) + T(v)$ and $T(\alpha v) = \alpha\, T(v)$ for all $u, v \in V$ and $\alpha \in \mathbb{F}$.

   (b) A *subspace* of a vector space V is ...

   Solution: A nonempty subset $U \subseteq V$ that

   - contains the zero element 0_V of the vector space V;
   - closed under addition, i.e., if $u, v \in U$, then $u + v \in U$;
   - closed under scalar multiplication, i.e., if $u \in U$ and $\alpha \in \mathbb{R}$, then $\alpha u \in U$.

   (c) To say that a list of vectors $(x_1, x_2, \ldots, x_d)$ in a vector space X is *linearly dependent* means ...

   Solution: There exist scalars $a_1, \ldots, a_d \in \mathbb{F}$, not all zero, such that

   $$a_1 x_1 + \cdots + a_d x_d = 0_X.$$

   Equivalently, at least one x_j can be written as a linear combination of the others.

2. Let $\vec{v}_1, \ldots, \vec{v}_m$ be vectors in $\mathbb{R}^n$. You are not allowed to use the word *dimension* when doing this problem.

   (a) Prove that if $m > n$, then $(\vec{v}_1, \ldots, \vec{v}_m)$ is not linearly independent.

   Solution: Form the $n \times m$ matrix $A = [\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_m]$ with the $\vec{v}_j$ as columns. Consider the homogeneous system

   $$A\vec{c} = \vec{0}, \qquad \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}.$$

[*] For full credit, please write out fully what you mean instead of using shorthand phrases.

In row-reduced echelon form, there are at most n pivot columns (since there are n rows), so when $m > n$ there are at least $m - n \geq 1$ free variables. Hence $A\vec{c} = \vec{0}$ has a nonzero solution $\vec{c} \neq \vec{0}$. Writing this out,

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$$

with not all c_j equal to 0, which is a nontrivial linear relation. Therefore $(\vec{v}_1, \ldots, \vec{v}_m)$ is not linearly independent.

(b) Prove that if $m < n$, then $(\vec{v}_1, \ldots, \vec{v}_m)$ does not span $\mathbb{R}^n$.

Solution: Again let $A = [\vec{v}_1 \ \cdots \ \vec{v}_m]$ (an $n \times m$ matrix). If the list spanned $\mathbb{R}^n$, then for every $\vec{b} \in \mathbb{R}^n$ the system $A\vec{c} = \vec{b}$ would be consistent.

Row-reduce A to its RREF. With only m columns, there can be at most m pivot rows. When $m < n$, there is at least one row of zeros in the RREF of A. Choose $\vec{b}$ whose entry in that zero row position is nonzero (and arbitrary elsewhere). Then, in the augmented matrix, the last row reads $[\,0 \ \cdots \ 0 \mid b_*\,]$ with $b_* \neq 0$, which is inconsistent. Hence there exists $\vec{b} \in \mathbb{R}^n$ not expressible as a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$. Therefore the list does not span $\mathbb{R}^n$.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) The dimension of $\mathbb{C}$ is three.

Solution: FALSE. As a vector space over $\mathbb{R}$, a basis is $\{1, i\}$, so the size of a basis is 2.

(b) The subset of $\mathcal{P}_4$ defined by $\{f(x) \mid f(\pi) = 0\}$ has dimension five.

Solution: FALSE. Evaluation at π,

$$E : \mathcal{P}_4 \to \mathbb{R}, \qquad E(f) = f(\pi),$$

is a linear map and is not the zero map (e.g. $E(1) = 1 \neq 0$). Its kernel is exactly $\{f \in \mathcal{P}_4 : f(\pi) = 0\}$, which is a proper subspace of $\mathcal{P}_4$. Since $\mathcal{P}_4$ has 5 coefficients, the kernel cannot be all of $\mathcal{P}_4$; in fact one linear condition reduces the number of free coefficients by one, so the kernel has 4 degrees of freedom. Concretely, every $f \in \ker E$ can be written as

$$f(x) = (x - \pi)\, q(x), \qquad q \in \mathcal{P}_3,$$

which shows the kernel is in bijection with $\mathcal{P}_3$ (four coefficients). Thus the subset has dimension 4, not 5.

Alternatively, $\{x - \pi, (x - \pi)^2, (x - \pi)^3, (x - \pi)^4\}$ is a basis of the subset of dimension 4.